CCFU Proof 3

Conformal Preservation $J(0)^\top G\, J(0) = \beta\, G$

Given. The nonlinear memory map on $\mathbb{C}^2 = \mathbb{R}^4$:

$$F(z,w) = (z^2 + \beta w,\ z), \qquad \beta \in \mathbb{R},\ \beta \neq 0.$$

Write $z = x_1 + ix_2$, $w = x_3 + ix_4$.

Jacobian at the origin ($z = 0$, $w = 0$):

$$J(0) = \begin{pmatrix} 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \beta \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Bilinear form. Define $Q = x_1x_3 + x_2x_4$, represented by:

$$G = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad \text{sig}(G) = (2,2).$$

Note: under the convention $v^\top G\, v$, this matrix represents $2Q$. The scalar factor does not affect signature or conformal preservation.

Claim. $J(0)^\top G\, J(0) = \beta\, G$ for all $\beta \in \mathbb{R}$, $\beta \neq 0$.

Proof.

$$J(0)^\top = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \beta & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \end{pmatrix}.$$

$$J(0)^\top \cdot G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix}.$$

$$\big(J(0)^\top \cdot G\big) \cdot J(0) = \begin{pmatrix} 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \beta \\ \beta & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \end{pmatrix} = \beta\, G. \qquad \blacksquare$$

Signature. The eigenvalues of G are $+1, +1, -1, -1$, so $\text{sig}(G) = (2,2)$. Since $\text{sig}(2,2)$ is symmetric under sign reversal, $\beta < 0$ preserves the signature class.

Limitations.

- $\beta = 0$ gives $0 \cdot G$ (degenerate). The result holds for $\beta \neq 0$.
- This is conformal preservation *at the fixed point.*
- The full nonlinear map does not preserve Q globally:

$$Q(F(z,w)) = |z|^2\, \text{Re}(z) + \beta\, Q(z,w) \neq \beta\, Q(z,w) \text{ in general.}$$

Status.

[Proven] $J(0)^\top G\,J(0) = \beta\,G$, $\beta \in \mathbb{R}$, $\beta \neq 0$.

[Disproven for this Q] Global nonlinear preservation $Q(F(z,w)) = \beta\,Q(z,w)$.

[Resolved later — Proof 15] The differential signature is globally preserved: $\mathrm{sig}(J(z)^\top G\,J(z)) = (2,2)$ for all z, $\beta \neq 0$.

[Open] Whether a modified global form-level invariant exists.